Mr. R. S. Underwood proposes and answers, in the second discussion, an interesting question concerning trigonometric functions: when will the sine or cosine of an angle rationally expressible in degrees be rational? It may be remarked that it is well known that $\sin \theta$ and θ, when θ is expressed in radians, can never simultaneously be even algebraic, much less rational numbers, for $\theta \neq 0$, and that a similar statement holds for each of the other elementary trigonometric functions. Proofs of these theorems are somewhat intricate and are closely related to the famous proofs of the transcendentality of e and π. With the same notation, Mr. Underwood's theorem may be said to deal with the simultaneous rationality of θ and $\sin \pi\theta$. This belongs to a simpler order of ideas and may be settled by an easy and direct method of attack. It has perhaps a more obvious interest for the field of collegiate mathematics than the more difficult question mentioned above. Mr. Underwood leaves open the nature of the result for the tangent and cotangent; without doubt the facts can be ascertaine sid bymilar methods in these cases.

I. A Theorem on Hypocycloids, by the Method of Circular Coördinates.

By T. L. Bennett, University of Illinois.

Let the plane be referred to rectangular coördinates (X, Y). The circular coördinates (x, y) of the point (X, Y) are defined to be $x = X + iY$, $y = X - iY$.

A complex number $a + ib$, for which $a^2 + b^2 = 1$, is called a *turn*. A turn may be written $e^{i\theta}$, where $\theta = \cos^{-1} a = \sin^{-1} b$. It is easily shown that the conjugate of a turn is its reciprocal, and that any product of turns is a turn. All points which have turn coördinates lie on a unit circle about the origin. A turn will be denoted by some form of the letter t.

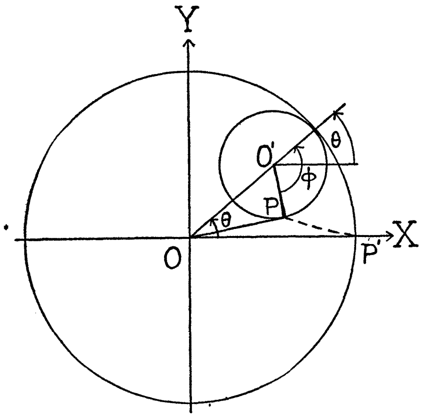
A curve may be represented by a map equation, $x = f(t)$. When t assumes turn values we get values of x, thus determining the points of the curve, because y is known as soon as x is known, being the conjugate of x. If $f(t)$ is a rational function, there is thus established a definite correspondence between the points of the unit circle and the points of the curve. If in the map equation t is replaced by any function of t which is a turn when t assumes turn values, the curve is not changed, as we have merely a different distribution of the parameter along the curve. With every map equation, $x = f(t)$, is associated another equation, namely, $\bar{x} = y = \overline{f(t)}$. If t be eliminated between these two equations, the result is the equation of the curve in circular coördinates.

The equation in circular coördinates of any straight line may be written in the form $t_1 x + y = rt_1$. The coefficient t_1 is called the *clinant* of the line, and determines the direction of the line thus: the angle which the line makes with the positive real axis is found to be

$$\phi = \frac{-1}{2i} \log(-t_1).$$

If a curve is defined by the map equation $x = f(t)$, it can be shown that the clinant of the tangent at any point is $-(dy/dt)/(dx/dt)$.

Let OX and OY be the real and imaginary axes, respectively, and let the circle O', of radius r_2, roll inside the circle O, of radius r_1, with $r_1 > r_2$, and r_1/r_2 rational. Any point on the circle O', as P, will describe a hypocycloid. Let the initial position of P be P', on OX. The map equation of the locus of P is obtained as follows:



$$\text{vector } OP = \text{vector } OO' + \text{vector } O'P,$$

$$x = (r_1 - r_2)e^{i\theta} + r_2 e^{i(\theta-\phi)}.$$

Since $r_1\theta = r_2\phi$, this equation becomes

$$x = (r_1 - r_2)t + r_2 \cdot \frac{1}{t^{\frac{r_1-r_2}{r_2}}},$$

where $t = e^{i\theta}$.

As a consequence of this equation it easily follows that if $r_1/r_2 = p/q$, the cycloid so generated will be identical with the cycloid for which $r_1/r_2 = p/(p-q)$. This means that if a circle of radius $r_1 - r_2$ be rolled inside the circle O, with the initial point of contact at P', the hypocycloid so generated will be identical with that generated by P.

If $r_1/r_2 = 2/1$, we have $x = r_2[t + (1/t)]$, which is the equation of the line segment of length $4r_2$ lying along the real axis, having its center at the origin. It will be found convenient to extend this equation as follows: if a and b are any complex numbers, then from the graphical addition and multiplication of complex numbers the following facts are obvious: $x = a + b[t + (1/t)]$ is the map equation of a line segment with center at $x = a$, of length $4|b|$, inclined to the positive real axis at the angle amp b, with extremities at the points for which t equals 1 and -1.

We shall now demonstrate the following theorem: *For any odd prime p, the $\frac{1}{2}(p-1)$ distinct p-cusped hypocycloids with common vertices may be arranged in cycles, so that each is the envelope of a chord of constant length taken upon the succeeding curve of the cycle.*

Let $r_1/r_2 = p/q$. Since p is prime, and $q < p$, we may take $r_1 = p$, and $r_2 = q$. Now as p remains fixed, and q takes on all integral values from 1 to $p - 1$ inclusive, there are generated $p - 1$ cycloids. But from a preceding theorem it is seen that only half of them are distinct, and these may be generated by q taking the values from 1 to $\frac{1}{2}(p-1)$ inclusive. Hence we shall further restrict q so that $q < \frac{1}{2}p$. The cycloid for which $q = k$ will be called, for brevity, the curve $q = k$.

The general curve of the system is

$$x = (p-q)t + \frac{q}{t^{\frac{p-q}{q}}} = (p-q)T^q + \frac{q}{T^{p-q}},$$

where $t = T^q$. For any particular value of q, as k, this equation is

$$x = (p-k)T^k + \frac{k}{T^{p-k}}.$$

Consider the curve[1]

$$x = T_1^k(p - 2k) + kT_1^{k-1}T + \frac{k}{T_1^{p-k-1}T}.$$

It is clear that it cuts the curve $q = k$ at the point $T = T_1$. But by computing the values of $-(dy/dt)/(dx/dt)$ for this new curve and the curve $q = k$, it is seen that the curves are tangent at the point $T = T_1$, because at this point the clinants of the two curves are the same. This new equation may be written

$$x = T_1^k(p - 2k) + \frac{k}{T_1^{\frac{p}{2}-k}}\left[T_1^{\frac{p-2}{2}}T + \frac{1}{T_1^{\frac{p-2}{2}}T}\right].$$

Hence this is the equation of a line segment of length

$$4\left|\frac{k}{T_1^{\frac{p}{2}-k}}\right| = 4k,$$

with extremities given by $T_1^{(p-2)/2}T = \pm 1$, that is, at the points

$$x = T_1^k(p - 2k) \pm \frac{2k}{T_1^{\frac{p}{2}-k}}.$$

These two points lie on the hypocycloid

$$x = (p - 2k)T^{2k} + \frac{2k}{T^{p-2k}}, \quad [\text{for } T = \pm \sqrt{T_1}],$$

which is either the equation of the curve $q = 2k$, or the equation o $q = p - 2k$, according as $2k$ is or is not less than $\frac{1}{2}p$.

Hence, if k be any positive integer less than $\frac{1}{2}p$, if a segment of length $4k$ be suitably taken on the tangents to the curve $q = k$, the extremities of these tangents lie either on the curve $q = 2k$ or on the curve $q = p - 2k$, as above indicated. It is then clear that the successive values of q for the sequence of curves of the theorem must be obtained by successive doubling,[2] by then reducing mod p to values between $-\frac{1}{2}p$ and $\frac{1}{2}p$, and then taking the absolute value of the result.

That the curves may be arranged in cycles follows from Fermat's Theorem: $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$; whence $k2^{(p-1)/2} \equiv \pm k \pmod{p}$. There will be one or more cycles according as $\frac{1}{2}(p-1)$ is or is not the smallest integer which will satisfy the congruence $k2^x \equiv \pm k \pmod{p}$. For example, if $p = 13$, the values of q form the single cycle 4 5 3 6 1 2, while for $p = 17$ we get the two cycles 3 6 5 7 and 8 1 2 4.

The proof for $p = 5$ is included in the foregoing as a special case.

[1] Professor Morley has called such curves "penosculants" of the original curve. See his paper "On the metric geometry of the plane n-line," *Trans. of the Amer. Math. Soc.*, vol. 1, p. 102.

[2] In the general equation of the hypocycloid it was assumed that $r_1 > r_2$. It may be readily shown that if $r_1 < r_2$, this equation represents an epicycloid, which is identical with the epicycloid generated by rolling a circle of radius $r_2 - r_1$ on the outside of the circle O, with the initial point of tangency at P'. Therefore the above analysis shows that if we start with any positive integer for q, (prime to p), then by successive doubling of q, without reducing mod p, we obtain an infinite sequence of epicycloids having the property mentioned in the theorem.

II. On the Irrationality of Certain Trigonometric Functions.

By R. S. Underwood, Purdue University.

Theorem: *If an angle, expressed in degrees, is rational and not a multiple of* $30°$, *its sine, cosine, secant, and cosecant are irrational.*

By De Moivre's Theorem,

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n. \tag{1}$$

Equating $\cos n\theta$ to the real part of the binomial expansion, we get

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{2!} \cos^{n-2} \theta \sin^2 \theta + \frac{n(n-1)(n-2)(n-3)}{4!} \cos^{n-4} \theta \sin^4 \theta - \cdots. \tag{2}$$

(The letters m and n stand for integers throughout this discussion. For convenience we shall call an angle integral, rational, or irrational in accordance with the character of its value expressed in degrees.)

From (2) it appears that $\cos n\theta$ can always be expressed in terms of integral powers of $\cos \theta$, and that $\cos \theta$ is irrational if $\cos n\theta$ is irrational. Since $\cos (45° \pm m180°)$ is irrational, $\cos \dfrac{m180° \pm 45°}{45}$, or $\cos (4m \pm 1)°$ is irrational. Every odd number can be expressed in the form $(4m \pm 1)$, and therefore the cosine of every odd integral angle is irrational.

Testing the equation

$$\sin 30° = \sin 10°(3 - 4 \sin^2 10°) = 1/2$$

for rational roots, we find that $\sin 10°$ is irrational. Since $\cos 30°$ and hence $\cos 10°$ is irrational, it follows that the cosines of $(100° \pm m180°)$, $(80° \pm m180°)$, and $(40° \pm m180°)$, and hence of $(9m \pm 5)°$, $(9m \pm 4)°$, and $(9m \pm 2)°$, are irrational. The last angle includes the form $(9(2s) \pm 2)°$; therefore, $\cos (9m \pm 1)°$ is irrational. The only even numbers which cannot be expressed in these forms are multiples of 6. By a geometric method we can show that $\sin 18° = (\sqrt{5} - 1)/4$ and $\cos 18° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$ are irrational, and hence this is true also of the cosines of 108°, 72°, 54°, 36°, 24°, 12° and 6°. Then, since $\cos 12°$ is irrational, $\sin 6°$ (or $\sqrt{[1 - \cos 12°]/2}$) is irrational; and we can add the cosines of 96°, 84°, 48°, 42°, $66° = (180° - 48°)/2$, and $78° = (180° - 24°)/2$ to the list. Thus, in general, when n is not a multiple of 30°, $\cos n°$, and obviously $\sin n°$ as well, is irrational.

Furthermore, $\cos (m°/n)$ when m/n is a rational non-integral fraction reduced to its lowest terms and m is not a multiple of 30°, is evidently irrational since $\cos m°$ is irrational. For every such angle $m°/n$ there is a complementary angle $h°/n$ $(h \neq k30)$ whose sine is irrational, and vice versa; hence both the sine and the cosine of the general rational angle $m°/n$ $(m \neq k30)$ are irrational.

To investigate an angle of the form m/n when m is a multiple of 30 and prime

to n (which is therefore odd), we may use the following expansion of $\sin n\theta$, valid when n is odd:[1]

$$\sin n\theta = n \sin\theta - \frac{n(n^2-1^2)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta - \cdots (-1)^{(n-1)/2}2^{n-1}\sin^n\theta. \tag{3}$$

The coefficients in this expansion are integers, as may be seen by equating the imaginary parts in the expansion of (1).

The coefficient of the $(r+1)$th term in (3) contains the factor 2^{2r}. For, upon writing this coefficient,

$$\frac{(n-2r+1)\cdots(n-3)(n-1)n(n+1)(n+3)\cdots(n+2r-1)}{(2r+1)!},$$

we see that in the $2r$ successive even numbers contained in the numerator, there will be

$2r$ numbers containing the factor 2,
r numbers containing the factor 2^2,
at least $[r/2]$ numbers containing the factor 2^3,
at least $[r/2^2]$ numbers containing the factor 2^4,
. . . .
at least $[r/2^\mu]$ numbers containing the factor $2^{\mu+2}$.

By $[r/2]$ is meant the largest integer in $r/2$. Choose μ so that $[r/2^\mu] = 1$. The sum of the exponents of 2 in the numerator will be *at least*

$$2r + r + \left[\frac{r}{2}\right] + \left[\frac{r}{2^2}\right] + \cdots + \left[\frac{r}{2^\mu}\right].$$

In $(2r+1)!$ the sum of the exponents of 2 is *exactly*

$$r + \left[\frac{r}{2}\right] + \left[\frac{r}{2^2}\right] + \cdots + \left[\frac{r}{2^\mu}\right].$$

Hence the excess of exponents of 2 in the numerator will be at least $2r$, and the coefficient will contain the factor 2^{2r}.

Letting $2r+1 = n$, we see that the exponent of 2 in the coefficient of $\sin^n\theta$ is no greater than the required $2r$. Then we can write

$$2^{n-1}\sin^n\theta - 2^{n-3}k_1\sin^{n-2}\theta + 2^{n-5}k_2\sin^{n-4}\theta - \cdots \pm n\sin\theta = \pm\sin n\theta.$$

If we put

$$x = 2\sin\theta,$$

then

$$x^n - k_1x^{n-2} + k_2x^{n-4} - \cdots \pm nx = \pm 2\sin n\theta.$$

All the coefficients in this equation except the absolute term on the right are integers. Let $\theta = m^\circ/n$, where m is an odd multiple of 30 or a multiple of 180, and prime to n. Then $2\sin n\theta = 2\sin m^\circ = 0, \pm 1, \pm 2$, and the absolute term is also an integer. Therefore the rational roots are integers, and $\sin\theta$ is half of an integer. Then $\sin\theta = 0, \pm\frac{1}{2}$, or ± 1, which is impossible.

[1] Loney: *Analytical Trigonometry*, p. 69.

This disposes of every case except when m is a multiple of 60 but not of 180. In this case $\sin n\theta = \pm \frac{1}{2}\sqrt{3}$, and by (3), $\sin \theta$ is irrational.

It has now been shown that the sine of every rational angle not a multiple of 30° is irrational. Obviously this establishes the theorem for the cosine, secant, and cosecant functions as well.

An analogous theorem, differing only in the substitution of 45° for 30°, may probably be shown to hold true for the tangent and cotangent.

RECENT PUBLICATIONS

REVIEWS

AMERICAN MEN OF SCIENCE.

American Men of Science. A Biographical Directory. Edited by J. McK. CATTELL and D. R. BRIMHALL. Third edition. Garrison, N. Y., The Science Press, 1921. 4to. 8 + 808 pp. Price $10.00.

The first edition of this work, published early in 1906, contained brief sketches of about 4,000 living American[1] men and women of science, the second edition, published late in 1910, about 5,500; the present edition contains some 9,500 sketches. The increase in the number of sketches measures roughly the increase of scientific workers.

In the first edition a star was prefixed to the subject of research in the case of one thousand of the sketches of students of the natural and exact sciences in the United States. In each of the twelve principal sciences[2] the names were arranged in the order of merit by ten leading students of the science. In this way the subjects of research of 80 mathematicians and 50 astronomers were starred.

In the second edition, the thousand leading men of science were determined in the same manner as in the first edition, stars being added to the subjects of research in the case of 269 new men; of these 20 were mathematicians. Their names, the names of the original 80 mathematicians, as well as of the 29 new mathematicians whose subjects of research are starred in the present edition, are given below.

It will be observed that in all editions the total number of sketches, in connection with which mathematics is starred, is 129. Of these, 11 refer to those who have died. There remain 118 living mathematicians with a star in the present edition, since a star once given in a sketch is not removed in subsequent editions. Hence 38 of these mathematicians are now no longer regarded as among the 80

[1] This term is interpreted as applying not only to natives of the United States and of the Dominion of Canada, but also to foreigners temporarily resident in these countries; for example: O. Bolza and P. Boutroux.

[2] Mathematics, physics, chemistry, astronomy, geology, botany, zoölogy, physiology, anatomy, pathology, anthropology, and psychology.